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NATURAL CONVECTION OF AN ELECTRICALLY CONDUCTING FLUID IN THE PRESENCE OF A MAGNETIC FIELD*

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Abstract—The case of a vertical hot plate surrounded by an electrically conducting fluid is examined in the presence of a magnetic field acting in a direction perpendicular to the induced movement caused by the buoyant forces.

It is found that similarity solutions exist, provided that the intensity of the magnetic field changes with the inverse fourth root of the distance measured in the direction of the flow.

The resulting differential equations of motion and energy have solutions depending on the Prandtl number, the Grashof number and a non-dimensional third number (say Z) representing the ratio of the ponderomotive force over the buoyant force.

Theoretical asymptotic solutions have been obtained for constant wall temperature in the following cases:

(a) Very high Prandtl and small Z numbers. In this case the inertia forces may be neglected.

(b) Very high Z numbers regardless of Prandtl numbers.

(c) Zero and small Prandtl numbers.

Exact solutions obtained by an analogue computer are also reported. It is found that the action of the magnetic field is to decelerate the flow thus decreasing the Nusselt number. For a constant Prandtl number the rate of decrease of the heat transfer coefficient with increasing values of the non-dimensional number Z is higher for smaller values of Z; on the other hand for the same value of the parameter Z, the rate of decrease of the same coefficient is higher for lower Prandtl numbers.

The case of non-similarity solutions is also investigated; the basic differential equations for a constant transverse magnetic field and fields depending on a power of the vertical distance are given.

It is found that experiments in the laboratory are feasible since the parameter Z is of the order of one to ten for liquid metals.

NOMENCLATURE

a_1, a_2, coefficients defined by equations (37) and (38);

B, magnetic inductance;

B_0, characteristic magnetic inductance defined in equation (9);

c, quantity defined by equation (5);

Gr, Grashof number defined by equation (6);

g, gravitational constant;

k, in equation (21) coefficient of thermal conductivity, in all other cases denotes $-\Theta_w'$;

L, characteristic length in the direction of main flow;

Nu, Nusselt number;

Pr, Prandtl number;

p, function defined by equation (16);

T, temperature;

u, velocity component in the x direction;

v, velocity component in the y direction;

x, Cartesian co-ordinate;

y, Cartesian co-ordinate;

Z, magnetic parameter defined by equation (10);

Z^*, parameter defined by equation (25).

Greek symbols

α, thermal diffusivity;

Γ, gamma function;

ζ, similarity function defined by equation (4);

η, similarity parameter defined by equation (4);

Θ, non-dimensional temperature function defined by equation (3);

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κ, κ', transformation constants defined by equations (A2) and (A6);
λ, λ', transformation constants defined by equations (A2) and (A6);
ν, kinematic viscosity;
ρ, density;
σ, electrical conductivity;
ψ, stream functions defined by equation (4).

Subscripts
e, denotes the edge of the boundary layer;
w, refers to the wall;
∞, refers at an infinite distance from the wall;
0, denotes new transformed variables in equation (A2);
Primes denote differentiation with respect to the variable η.

1. INTRODUCTION

In the present paper the case of a hot vertical plate surrounded by an electrically conducting fluid for constant wall temperature is examined in the presence of a magnetic field acting transversally to the induced movement caused by the buoyant forces. The geometry is similar to the classical Pohlhausen problem [1]. As in the case of the magnetic boundary layers around wedges investigated in [2], it is first examined whether similarity solutions for the natural convection case exist. This question will be answered in the affirmative; different asymptotic solutions of the appropriate similarity equations will be given.

The same general problem reported here, has been tackled independently in [3, 4, 5]. In [4] the only existing similarity solution in the case of the vertical wall was recognized, nevertheless a solution was given for a non-similarity case. In [5] the similarity solution in the steady state case is obtained by using an integral method in the manner of [6]; in the same reference the solution of the transient problem is also obtained. In [7] a class of similarity solutions is discussed by allowing non-isothermal conditions at the wall.

In the present work, use is made of Saunder's approximate method of integration [8]. An exact solution is also obtained through the use of an analogue computer and the results are compared with the available approximative solutions.

Asymptotic solutions are also presented for the following cases:

(a) High Prandtl numbers and small magnetic fields.
(b) High magnetic fields regardless of Prandtl numbers.
(c) Zero and small Prandtl numbers.

A class of non-similarity solutions is also discussed for which the basic differential equations resulting from a suitable series expansion scheme, are given.

Qualitatively, anticipating the solution, the following things are expected to happen: In the absence of an applied or induced electric field the magnetic lines offer resistance to the flow and as a result the thermally induced motion is retarded. This means that the shear stress at the wall is lowered and as a result the heat transfer coefficient will also be lowered.

Particular emphasis will be given to the solution for Prandtl numbers of the order 0·01, since both liquid metals and highly ionized gases are in this range. The geometry and flow conditions of the present paper may be approximated in the laboratory by making use of liquid metals. One perhaps may think of possible applications in the interior of the earth, or at the surface of the sun where steep temperature gradients exist in the presence of magnetic fields.

2. BASIC ASSUMPTIONS AND EQUATIONS

The following assumptions will be made in the present work:

(a) The magnetic field is constant in the direction perpendicular to the wall, throughout the thickness of the boundary layer. It remains also perpendicular to the oncoming stream lines.
(b) The induced electric current does not distort appreciably the applied magnetic field.
(c) The coefficient of electrical conductivity is a scalar and remains constant everywhere.

(d) The electric field calculated is the same frame of reference in which velocity is measured is zero.

Assumption (a) is not difficult to realize in practice if the magnetic field is created by the parallel poles of a strong electromagnet and if the flow takes place around the central region where the field will be most uniform. It should be emphasized however that this last region should be comfortably greater than the thickness of the boundary layer. If the uniformity of the applied magnetic field cannot be assumed, then one may use an average magnetic field calculated according to the specific law of its variation. In [9] it was shown in a similar case that the final result is equally acceptable.

Assumption (b) is true as long as the magnetic Reynolds number is small; for terrestrial dimensions with rather low velocities and electrical conductivities corresponding to liquid metals, this is always the case.

Assumption (c) should be understood to be valid as long as the electrical conductivity is calculated at an average temperature. In liquid metals it is always a scalar, whereas in ionized gases its scalar character will be conserved as long as the collision frequency of the particles is much higher than the cyclotron frequency.

Assumption (d) is exact as long as the induced current lines close in themselves; in a two dimensional case, like the one under consideration, this occurs at infinity. From a practical point of view it may easily be arranged by an appropriate short circuit in the plane x, z where x is the direction of the vertical plate, y the direction of the magnetic field with x and y mutually perpendicular. The induced current will appear in the z direction and according to Ohm's law after taking into account assumption (d) it will be equal to σuB. In this case the retarding force per unit volume $\mathbf{J} \times \mathbf{B}$ acting in the direction x will be equal to $-\sigma B^2 u$.

It will be furthermore assumed that all equilibrium and transport properties are constant.

The conservation equations of mass and momentum are then written as follows:†

† See [10] for details regarding these equations in the non-magnetic case. The nomenclature of this reference is used here as much as possible.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} + g\left(\frac{T_w - T_\infty}{T_\infty}\right)\Theta - \frac{\sigma u B^2}{\rho} \tag{2}$$

where

$$\Theta \equiv \frac{T - T_\infty}{T_w - T_\infty}. \tag{3}$$

The following definitions are now made:

$$\left.\begin{aligned} u &= \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x} \\ \eta &= \frac{cy}{\sqrt[4]{x}}, \ \psi = 4\nu c x^{3/4}\zeta(\eta) \end{aligned}\right\} \tag{4}$$

where

$$c = \sqrt[4]{\left(\frac{g(T_w - T_\infty)}{4\nu^2 T_\infty}\right)} = \frac{Gr^{1/4}}{\sqrt{(2)}x^{3/4}} \tag{5}$$

with

$$Gr = \frac{gx^3(T_w - T_\infty)}{\nu^2 T_\infty}. \tag{6}$$

In the above, ψ represents the stream function, $\zeta(\eta)$ a similarity function, η the similarity parameter and Gr the local Grashof number. With the help of the above it is easy to calculate that:

$$\left.\begin{aligned} u &= 4\nu x^{1/2} c^2 \zeta' \\ v &= \nu c x^{-1/4}(\eta\zeta' - 3\zeta). \end{aligned}\right\} \tag{7}$$

Equation (2) can now be transformed from the (x, y) system of co-ordinates to the (x, η) as follows:

$$\zeta''' + 3\zeta\zeta'' - 2(\zeta')^2 + \Theta - \frac{\sigma B^2 x^{1/2}}{\rho\nu c^2}\zeta' = 0. \tag{8}$$

The primes indicate differentiation with respect to the parameter η.

The last term in the above equation represents the ponderomotive term and it is seen that a similarity solution exists if it is independent of the distance x. This can only be done with a magnetic field varying with x as follows:

$$\frac{B}{B_0} = \left(\frac{x}{L}\right)^{-1/4} \tag{9}$$

The above condition insures that the non-dimensional ratio Z is independent of x and is given as follows:†

$$Z \equiv \frac{\sigma B^2 x^{1/2}}{\rho \nu c^2} = \frac{\sigma B_0^2 L^{1/2}}{\rho \nu c^2}. \tag{10}$$

From equation (9) one can see that there is a singular point at $x = 0$ (at the leading edge), nevertheless it is well known that the boundary layer equations are not valid there. The present situation is exactly the same with the problem of existence of similarity solutions in laminar flow with mass transfer at the wall where in the case of the flat plate it is demanded that the rate of mass injection be inversely proportional to the square root of x. Both types of singularity are integrable.

Using the magnetic parameter Z the equation of motion becomes:

$$\zeta''' + 3\zeta\zeta'' - 2(\zeta')^2 + \Theta - Z\zeta' = 0. \tag{11}$$

Next we write the energy equation by neglecting the viscous and Joulean dissipation. This is a customary simplification since both energies per unit volume are negligibly small when compared with the amount of energy conducted. In terms of the function Θ we have:

$$u\frac{\partial \Theta}{\partial y} + v\frac{\partial \Theta}{\partial x} = \alpha \frac{\partial^2 \Theta}{\partial y^2} \tag{13}$$

where α is the thermal diffusivity.

Using the similarity transformation the above becomes:

$$\Theta'' + 3(Pr)\zeta\Theta' = 0 \tag{14}$$

where Pr is the Prandtl number.

The boundary conditions to the basic equations (12) and (14) are as follows:

$$\left.\begin{array}{lll} \text{At } \eta = 0: & \zeta = \zeta' = 0, & \Theta = 1 \\ \eta = \infty: & \zeta' = 0, & \Theta = 0. \end{array}\right\} \tag{15}$$

This is a system of ordinary nonlinear differential equations of the fifth order. Their solution is well known for different Prandtl numbers in the non-magnetic case with $Z = 0$. The purpose of the remaining portion of this paper is the investigation of the influence of the parameter Z.

† In terms of other non-dimensional parameters it is easy to show that Z is the ratio of the Hartmann number M divided by the square root of the Grashof number. The Hartmann number is defined here as the ratio of the ponderomotive force over the viscous force, a more natural definition than the usual one. More precisely: $Z = 2M/(Gr)^{1/2}$.

3. SOLUTION OF THE SIMILARITY EQUATIONS

In order to obtain an approximate solution the method described by Saunders [8] will be used. We introduce a new function defined as follows:

$$p(\Theta) = \frac{d\Theta}{d\eta}. \tag{16}$$

From equation (14) we find

$$\zeta = -\frac{1}{3Pr}p' \tag{17}$$

where the prime on p denotes differentiation with respect to Θ. Furthermore we assume that the function p is given by a polynomial of the third degree as follows:

$$p(\Theta) = -\Theta_w'[(1-\Theta)^3 - 1]. \tag{18}$$

The above satisfies the boundary conditions at $\Theta = 0$ ($\eta \to \infty$) and $\Theta = 1$ ($\eta \to 0$).

The temperature gradient at the wall Θ_w', will be estimated by satisfying the equation of conservation of momentum at the value $\Theta = \frac{1}{2}$. Substitution of equation (18) in equation (11) gives:

$$-\Theta_w' = \sqrt{\left\{\frac{\frac{-7Z}{8Pr} + \sqrt{\left[\left(\frac{7Z}{8Pr}\right)^2 + 2\left(\frac{21}{4Pr} + \frac{7}{8Pr^2}\right)\right]}}{2\left(\frac{21}{4Pr} + \frac{7}{8Pr^2}\right)}\right\}}. \tag{19}$$

In another form we find

$$-\Theta_w' = (-\Theta_w')_{Z=0}\sqrt{\left\{\left(\frac{7Z}{8Pr}\right)(-\Theta_w')^2_{Z=0} + \sqrt{\left[\left(\frac{7Z}{8Pr}\right)^2(\Theta_w')^4_{Z=0} + 1\right]}\right\}}. \tag{20}$$

The equivalent result of [5] in which a parabolic non-dimensional temperature profile and an appropriate cubic velocity profile are used, is the following:

$$-\Theta'_w = \frac{1}{\sqrt{3}}\left(\frac{Pr}{\frac{20}{21}+Pr}\right)^{1/2}\left[\sqrt{\left(\frac{Z^2}{4}+\frac{12}{5}Pr+\frac{16}{7}\right)}-\frac{Z}{2}\right]^{1/2}. \quad (21)$$

An exact solution of the fundamental equations (11) and (14) was found by using an analogue computer. The functions $-\Theta'_w$ and ζ''_w are given in Table 1 for different values of the parameters Pr and Z. For the case $Z = 0$ and two Prandtl numbers the results obtained are compared with the ones presented in [11] by the use of a digital computer. The agreement is very good. For low Prandtl and high Z numbers it was difficult to obtain sound solutions through the analogue computer.

$$Nu = \frac{1}{\sqrt{2}}\sqrt{\left\{\frac{-\frac{7Z}{8Pr}+\sqrt{\left[\left(\frac{7Z}{8Pr}\right)^2+2\left(\frac{21}{4Pr}+\frac{7}{8Pr^2}\right)\right]}}{2\left(\frac{21}{4Pr}+\frac{7}{8Pr^2}\right)}\right\}}(Gr)^{1/4}. \quad (23)$$

For relatively small Prandtl numbers, equation (19) gives†

$$-\Theta'_w = \sqrt{\left(\left\{-\frac{Z}{2}+\sqrt{\left[\left(\frac{Z}{2}\right)^2+\frac{4}{7}\right]}\right\}(Pr)\right)}. \quad (24)$$

Table 1

Z	$Pr = 0{\cdot}73$		$Pr = 0{\cdot}30$		$Pr = 0{\cdot}10$		$Pr = 0{\cdot}03$		$Pr = 0{\cdot}02$		$Pr = 0{\cdot}01$	
	$-\Theta'_w$	ζ''_w	$-\Theta'_w$	ζ''_w	$-\Theta_w$	ζ''_w	$-\Theta'_w$	ζ''_w	$-\Theta'_w$	ζ''_w	$-\Theta'_w$	ζ''_w
0	0·5076	0·6680	0·3598	0·7486	0·2284	0·8618	0·1332	0·9270	0·1112	0·9606	0·0804	0·9900
	Digital computer values from [11]						0·13464	0·93841	0·11164	0·95896		
0·5			0·3372	0·6816	0·2138	0·7670	0·1238	0·8164	0·1038	0·8444	0·0752	0·8646
1	0·4510	0·5677	0·3148	0·6206	0·1998	0·6804	0·1160	0·7272	0·0962	0·7504	0·0707	0·7658
3·0	0·3730	0·4472										
5·0	0·3456	0·3750										

In Fig. 1, equations (19) and (21) are compared with the analogue computer results. For low Prandtl and small Z numbers equation (19) is in better agreement with the exact solution when compared with equation (21). For high Z numbers however equation (21) is better than equation (19). This behavior will become clear when we discuss the asymptotic solutions.

The amount of heat transferred per unit area per unit time to the fluid is given as follows:

$$q(x) = -kcx^{-1/4}\,\Theta'_w\,(T_w - T_\infty). \quad (22)$$

Using the ordinary definition of Nusselt number in conjunction with equations (20) and (22) we calculate

Similarly equation (21) gives:

$$-\Theta'_w = \sqrt{\left(\frac{7}{20}\left\{-\frac{Z}{2}+\sqrt{\left[\left(\frac{Z}{2}\right)^2+\frac{16}{7}\right]}\right\}(Pr)\right)}. \quad (25)$$

From the above results we draw the following conclusions: For a constant Prandtl number the rate of decrease of the heat transfer coefficient is higher for smaller values of Z; on the other hand for the same value of the magnetic field (same Z) the heat transfer coefficient drops faster for smaller Prandtl numbers.

† For $Pr < 0{\cdot}02$, $21/Pr$ can be neglected compared to $7/8Pr^2$.

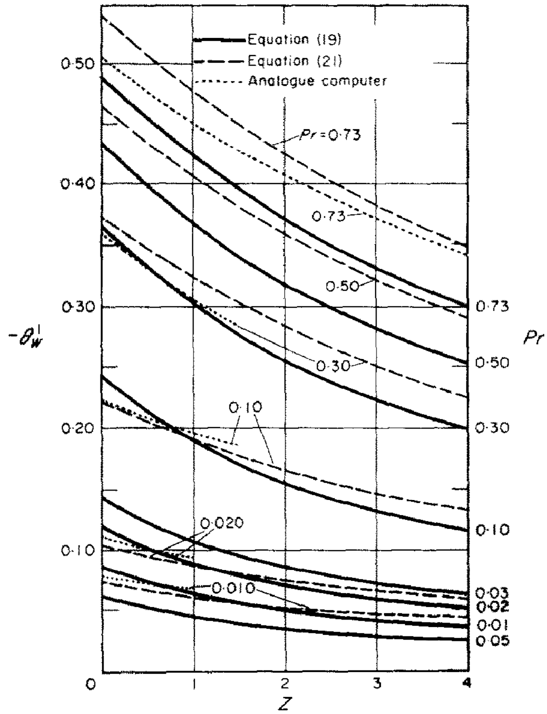

FIG. 1. Temperature gradients for different Prandtl numbers and magnetic numbers Z.

4. ASYMPTOTIC SOLUTIONS

Small Prandtl numbers

Let us now examine the case of very small Prandtl numbers when the viscous forces may be neglected. Furthermore let us assume that the buoyant force expressed by Θ in equation (11) is represented by an average value equal to $\frac{1}{3}$.† Neglecting the term $3\zeta\zeta''$ we have:

$$-2(\zeta')^2 + \tfrac{1}{3} - Z\zeta' = 0. \qquad (26)$$

From which

$$\zeta' = -\frac{Z}{4} + \sqrt{\left[\left(\frac{Z}{4}\right)^2 + \frac{1}{6}\right]} \equiv Z^* \qquad (27)$$

and

$$\zeta = Z^*\eta. \qquad (28)$$

From equation (28) it is seen that within the present approximation the term $3\zeta\zeta''$ is indeed negligible. Substitution of the above approximation found for ζ into the energy equation gives:

$$\Theta'' + 3(Pr)Z^*\eta\Theta' = 0. \qquad (29)$$

It is now easy to integrate this equation once and calculate the slope of Θ at the wall. We find:

$$-\Theta_w' = \frac{\sqrt{\frac{3}{2}}}{\Gamma(\frac{3}{2})}(Z^*Pr)^{1/2} \qquad (30)$$

or

$$-\Theta_w' = 1{\cdot}38\,(Z^*Pr)^{1/2}.$$

In the above Γ represents the gamma function. For $Z = 0$ we calculate

$$(-\Theta_w')_{Z=0} \approx 0{\cdot}88\,(Pr)^{1/2}. \qquad (31)$$

On the other hand for $Z = 0$ Saunder's formula [equation (19)] yields

$$(-\Theta_w')_{Z=0} = 0{\cdot}868\,(Pr)^{1/2}. \qquad (32)$$

In [11] it is quoted that the asymptotic solution for $Z = 0$ and $Pr \to 0$ found by Lefevre is given as follows:

$$(-\Theta_w')_{Z=0} = 0{\cdot}85\,(Pr)^{1/2}. \qquad (33)$$

The error of equations (31) and (32) is between 2 and 3·5 per cent and it is thus seen that our assumption on the average value of Θ for small Prandtl numbers is correct.

Equation (21) based on the results of [5] or [6] for $Z = 0$ gives

$$(-\Theta_w')_{Z=0} = 0{\cdot}727\,(Pr)^{1/2}. \qquad (34)$$

The error from the exact solution in this limiting case is about 14·5 per cent.

From the above it follows that the Saunders formula constitutes a better approximation for the smaller Prandtl numbers and moderate values of Z than Eckert's formula.

Substitution of equation (27) in equation (30) gives:

$$-\Theta_w' = \sqrt{\left(\left\{-\frac{Z}{2{\cdot}1} + \sqrt{\left[\left(\frac{Z}{2{\cdot}1}\right)^2 + 0{\cdot}6\right]}\right\}(Pr)\right)}. \qquad (35)$$

Comparison of the above with equation (24) derived from the Saunders approximation shows that they are in good agreement.

† This is the exact average for a parabolic non-dimensional temperature profile. See [6].

Let us now consider the case of small Prandtl numbers and high Z numbers. Then equations (24) and (35) give respectively the following asymptotic expressions:

$$-\Theta'_w = 0{\cdot}756 \sqrt{[(Pr)/Z]} \tag{36}$$

$$-\Theta'_w = 0{\cdot}794 \sqrt{[(Pr)/Z]}. \tag{37}$$

The difference being the two coefficients of proportionality is less than 5 per cent. The corresponding result from [5] is:

$$-\Theta'_w = 0{\cdot}895 \sqrt{[(Pr)/Z]} \tag{38}$$

From equation (29) we note however that for very high values of the parameter Z the value of Θ will tend in the limit to be one, rather than $\frac{1}{3}$. With this change in equation (26), equation (37) must be replaced with the following:

$$-\Theta'_w \approx 1{\cdot}38 \sqrt{[(Pr)/Z]}. \tag{39}$$

It is now clear that the exact asymptotic solution should yield a numerical coefficient between 0·794 and 1·38. As a matter of fact the average of these two values 1·087 coincides with the exact one which can be computed as follows:†

Neglecting both the inertia and viscous forces in the equation of motion (11) we have:

$$\Theta = Z\zeta'.$$

Substitution in the energy equation (14) yields:

$$\zeta''' + 3Pr\zeta\zeta'' = 0. \tag{40}$$

The boundary conditions are:

$$\zeta(0) = 0, \quad \Theta(0) = Z\zeta'(0) = 1, \quad \zeta'(\infty) = 0.$$

The above differential equation has been solved numerically in [12, 13]. The terms of the present notation the result is:

$$-\Theta'_w = 1{\cdot}087 \sqrt{[(Pr)/Z]}. \tag{41}$$

It is concluded therefore that equations (37) and (38) are in error of about 30 and 18 per cent respectively.

From equation (30) one sees that in effect the magnetic field lowers the heat transfer coefficient by simply lowering the Prandtl number by the factor Z^*. This behavior of the magnetic field is not new. In [2], where the similarity solutions for magnetic boundary layers in wedge type of flows are discussed, the same dependence was shown to exist. One is tempted, for purposes of calculating heat transfer coefficients to define an equivalent magnetic Prandtl number from relations of the form:

$$(Pr)_{\text{magn.}} = f(Pr, Z).$$

Zero Prandtl number and zero magnetic field

It is shown in the Appendix that the velocity profile for the case of $Z = 0$ with $Pr \to 0$ and $\Theta'_w \to 0$ corresponds exactly to an incompressible similarity solution for a wedge of an angle equal to $2\pi/3$. This asymptotic solution seems to have escaped the attention of workers in the field. It is worth noting that the value of $\zeta''(0)$ for $Pr \to 0$ is equal to 1·070 as found in the Appendix. The lowest calculated value of $\zeta''(0)$ for $Z = 0$, is the one of [11] obtained by numerical integration and for $Pr = 0{\cdot}003$ is equal to 1·0223.

Large Prandtl numbers

An attempt will now be made to calculate the heat transfer coefficient for very large Prandtl numbers. Obviously this solution can be only of theoretical interest, since for liquid metals or ionized gases the Prandtl numbers are smaller than one.

It is known that for highly viscous fluids the function Θ assumes the value zero for small values of the similarity parameter η the slope of Θ being very steep at the origin.

We make the following assumption

$$\Theta = \exp(-k\eta)\,. \tag{42}$$

The coefficient k is a numerical constant which actually denotes the slope of Θ at the wall. By neglecting the inertia forces, the equation of motion yields:

$$\zeta''' + \exp(-k\eta) - Z\zeta' = 0. \tag{43}$$

The solution of this equation satisfying the boundary conditions at the wall is:

$$\zeta' = \frac{1}{Z - k^2}\{\exp(-k\eta) - \exp[-\sqrt{(Z)}\eta]\}. \tag{44}$$

For high Prandtl numbers the boundary layer terminates at a very small value of η, so that in

† For this computation the author is indebted to Dr. Andreas Acrivos of the University of California at Berkeley.

the vicinity of the wall we may expand the quantity exp $(-k\eta)$ into a series, and retain the first two terms. For moderate magnetic fields, at least up to the order $Z \sim k^2$, we may also expand the term exp $[-\sqrt{(Z)}\eta]$ into a series and retain the first two terms. The final result is:

$$\zeta' \approx \frac{\eta}{\sqrt{(Z)} + k}. \tag{45}$$

This formula must now be corrected to take into account the fact that at $\eta \to \infty$, ζ' actually goes to zero rather than infinity. For this purpose let us assume that the boundary layer thickness is equal to η_e and furthermore that the actual behavior of ζ' is a parabolic one between zero and η_e of the form:†

$$\zeta' = a_1\eta\left(1 - \frac{\eta}{\eta_e}\right). \tag{46}$$

Since it was shown that equation (45) is valid for small values of η, for which the square component of equation (46) is small, we set:

$$a_1 = \frac{1}{\sqrt{(Z)} + k}.$$

On the other hand, the accuracy of the value of the heat transfer coefficient depends solely on a correct estimate of ζ. We therefore set the condition that the area under the curve of equation (46) be equal to the area under the straight line

$$\zeta' = a_2\eta$$

where a_2 is a coefficient of proportionality to substitute the one of equation (45), giving a better average value within the boundary layer thickness. We set:

$$\int_0^{\eta_e} a_2\eta \, d\eta = \int_0^{\eta_e} \frac{\eta(1 - \eta/\eta_e)}{\sqrt{(Z)} + k} \, d\eta. \tag{47}$$

From the above we find a_2 and hence:

$$\zeta' = \frac{1}{3[\sqrt{(Z)} + k]}\,\eta. \tag{48}$$

A simple integration yields:

$$\zeta = \frac{1}{6[\sqrt{(Z)} + k]}\,\eta^2. \tag{49}$$

† This form is suggested by the known solution for ζ' in the case $Z = 0$.

Substitution of the above‡ into the energy equation gives the following value for $k = -\Theta'_w$ after a straightforward calculation:

$$-\Theta'_w = \frac{1}{(6)^{1/3}\,\Gamma(\frac{4}{3})}\left(\frac{Pr}{\sqrt{(Z)} + (-\Theta'_w)}\right)^{1/3} \tag{50}$$

or

$$-\Theta'_w = 0{\cdot}617\left(\frac{Pr}{\sqrt{(Z)} + (-\Theta'_w)}\right)^{1/3}.$$

We test now this result in the case where $Z = 0$ for which a solution is known (9). We find:

$$(-\Theta'_w)_{Z=0} = 0{\cdot}695\, Pr^{1/4}. \tag{51}$$

The coefficient of proportionality for Prandtl numbers 100 and 1000 is given in [10] as 0·690 and 0·692 respectively.§ The result is of equation (51) is therefore in good agreement with the exact solution (for $Z = 0$) and justifies our assumptions.

Equation (50) has been plotted in Fig. 2 for a value of Prandtl number equal to 100. It is seen that the heat transfer coefficient drops fast for small Z's and much slower later. In other words the same behavior is shown for high and low Prandtl numbers. Physically such a situation ought to be expected, since the ponderomotive force is proportional to the velocity; for the same degree of deceleration, corresponding to the same Prandtl number, higher increments of Z's are needed at the high Z's (where the local velocities are small) to produce the same drop in heat transfer with the one corresponding to an increment of ΔZ in the neighborhood of a lower Z.

Equation (50) is strictly speaking valid for small values of the parameter Z, of the order of $(-\Theta'_w)^2$ as explained before. For very high magnetic fields tending to make Θ'_w equal to

‡ The reason for seeking to estimate ζ as a power of η is explained by the fact that for the calculation of θ'_w we need the value of the definite integral

$$\int_0^\infty \exp\left(-3Pr \int_0^\eta \zeta \, d\eta\right) d\eta$$

which reduces to a gamma function for $\zeta \sim \eta^m$. Numerical calculations are needed when ζ is given by a polynomial exponential function.

§ In reality [10] gives the values corresponding to the mean heat transfer, which are the above divided by the factor $\frac{4}{3}\sqrt{2}$.

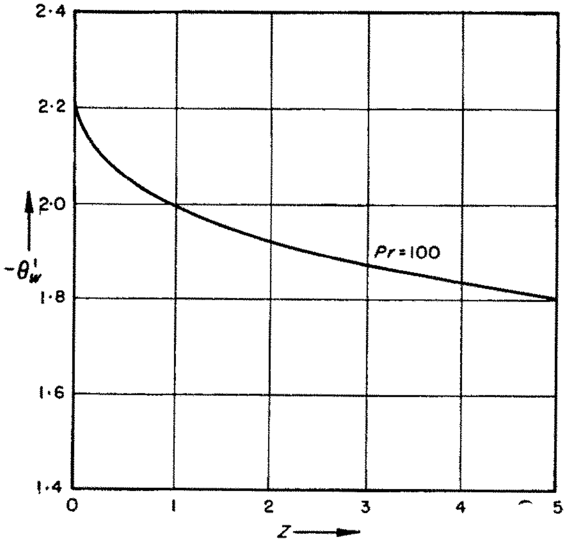

FIG. 2. Temperature gradients for $Pr = 100$ and different magnetic numbers Z.

zero, the viscous forces will be negligible and the heat transfer coefficient will exhibit the same behavior with equation (41) found for the case of very small Prandtl numbers.

In conclusion we now make an estimate of the order of magnitude of the parameter Z in order to assess whether an experiment is feasible in the laboratory in the course of which reductions in heat transfer will be observed. For liquid mercury with a magnetic field of 10 000 Gauss, a heating ratio $(T_w - T_\infty)/T_\infty = 1{\cdot}5$ and a characteristic length of 0·1 m, we find $Z \approx 1{\cdot}25$. If we choose a liquid metal lighter than mercury with a density ten times less, with all other conditions remaining the same, Z will be equal to 12·5. From Fig. 1 we estimate that for $Pr = 0{\cdot}02$ and $Z = 1$ a reduction in heat transfer, of 25 per cent should be expected, whereas for $Z = 5$ the reduction will be equal to 60 per cent.

It is concluded that laboratory experiments should be able to show the influence of a magnetic field in heat transfer in natural convection.

5. NON-SIMILARITY SOLUTIONS

For a magnetic field which does not vary according to the one fourth power law of equation (9), similarity in the sense of equation (11) does not exist and we must work with equation (8) instead.

Let us assume in the notation of [4] the following series expansion for the stream function ψ:

$$\psi = 4\nu c x^{3/4}\{\zeta_0(\eta) + (nx)^p\zeta_1(\eta) + (nx)^{2p}\zeta_2(\eta) + \ldots\} \quad (52)$$

For the Θ function we assume:

$$\Theta = \Theta_0(\eta) + (nx)^p\Theta_1(\eta) + (nx)^{2p}\Theta_2(\eta) + \ldots \quad (53)$$

Substitution of equations (52) and (53) into the momentum and energy equations yields the following results: in the zeroth approximation we get the equations of the non-magnetic case already solved in [1]. In the first approximation we get the following set:

$$\zeta_1''' + 3\zeta_0\zeta_1'' - 4(p+1)\zeta_0'\zeta_1' + (4p+3)\zeta_1\zeta_0 - \zeta_0' + \Theta_1 = 0 \quad (54)$$

$$\Theta_1'' + Pr\,\{3\zeta_0\Theta_1' - 4p\Theta_1\zeta_0' + (4p+3)\zeta_1\Theta_0'\} = 0. \quad (55)$$

Under the condition:

$$\rho\nu c^2(nx)^p = \sigma B^2 x^{1/2}. \quad (56)$$

Now let

$$\frac{B}{B_0} = \left(\frac{x}{L}\right)^\lambda. \quad (57)$$

The condition (56) becomes:

$$Z \equiv \frac{\sigma B_0^2 L^{1/2}}{\rho\nu c^2} = (nx)^p\left(\frac{L}{x}\right)^{2\lambda+\frac{1}{2}}. \quad (58)$$

Hence it is necessary for the power law (58) to have

$$p = 2\lambda + \tfrac{1}{2} \quad (59)$$

and

$$(nx)^p = Z\left(\frac{x}{L}\right)^p \quad (60)$$

In terms of the above, equations (52) and (53) become:

$$\psi = 4\nu c x^{3/4}\left\{\zeta_0(\eta) + \left[Z\left(\frac{x}{L}\right)^{2\lambda+\frac{1}{2}}\right]\zeta_1(\eta) + \left[Z\left(\frac{x}{L}\right)^{2\lambda+\frac{1}{2}}\right]^2\zeta_2(\eta) + \ldots\right\} \quad (61)$$

$$\Theta = \Theta_0(\eta) + \left[Z\left(\frac{x}{L}\right)^{2\lambda+\frac{1}{2}}\right]\Theta_1(\eta) + \ldots \quad (62)$$

It is obvious that the value $\lambda = -\frac{1}{4}$ yields the similarity case already discussed at length. Ref. 4 presents a numerical calculation for the case of $\lambda = \frac{1}{4}$ with $p = 1$. Equations (54) and (55) become:

$$\zeta_1''' + 3\zeta_0\zeta_1'' - 8\zeta_0'\zeta_1' + 7\zeta_1\zeta_0'' - \zeta_0' + \Theta_1 = 0 \quad (63)$$

$$\Theta_1'' + Pr\,\{3\zeta_0\Theta_1' - 4\Theta_1\zeta_0' + 7\zeta_1\Theta_0'\} = 0. \quad (64)\dagger$$

Of interest to a future investigator might be the case of a constant magnetic field corresponding to the case $p = \frac{1}{2}$.

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REFERENCES

1. E. Pohlhausen, Der Wärmeaustausch Zwischen Festen Körpen und Flüssigkeiten mit kleiner Reibung und kleiner Wärmeleitung, *Z. Angew. Math. Mech.* **1**, 115 (1921).
2. P. S. Lykoudis, On a class of compressible laminar boundary layers with pressure gradient for an electrically conducting fluid in the presence of a magnetic field, *Proceedings, IXth International Astronautical Congress, Amsterdam*, pp. 168–170. Springer-Verlag, Berlin (1959).
3. P. S. Lykoudis, Natural convection of an electrically conducting fluid in the presence of a magnetic field, *School of Aeronautical Engineering, Purdue University, Report No. A*-60-1, 1960. Presented at the Tenth International Congress of Applied Mechanics held on 31 August to 7 September 1960, at Stresa, Italy.
4. Y. Mori, On a laminar free-convection flow and heat transfer of electrically conducting fluid on a vertical flat plate in the presence of a transverse magnetic field, *Trans. Japan Soc. Aero. Space Sci.* **2**, 22–26 (1959).
5. A. S. Gupta, Steady and transient free convection of an electrically conducting fluid from a vertical plate in the presence of a magnetic field, *Appl. Sci. Res.* **A9**, No. 5, 319–333 (1960).
6. E. R. G. Eckert, *Heat and Mass Transfer*, pp. 312–315, McGraw-Hill, New York (1959).
7. B. L. Reeves, Similar solutions of the free convection boundary layer equations for an electrically conducting fluid, *J. Amer. Rocket Soc.* **31**, 557–558 (1961).
8. O. A. Saunders, Natural convection in liquids, *Proc. Roy. Soc.* **A172**, 55–71 (1939).
9. P. S. Lykoudis, The matching of the viscid and inviscid regions for the stagnation magnetic flow, *J. Aero. Space Sci.* **26**, 315–316 (1959).
10. H. Schlichting, *Boundary Layer Theory*. McGraw-Hill, New York (1955).
11. E. M. Sparrow and J. L. Gregg, Details of exact low Prandtl number boundary-layer solutions for forced and for free convection, *NASA, Memo* 2-27-59E, February (1959).
12. A. Acrivos, On the rate of mass transfer from a gas to a moving liquid film, *Chem. Engng Sci.* **9**, 242–249 (1958).
13. G. I. Taylor, *Proc. Roy. Soc.* **A253**, 295 (1959).

† In [4] the term $-4Pr\Theta_1'\zeta_0$ is given instead of $-4Pr\Theta_1\zeta_0'$. The final numerical results of this reference will have to change somewhat when this error is corrected.

APPENDIX

Asymptotic Solutions for $Z = 0$ and $Pr \to 0$

At $Pr \to 0$ we expect to have $\Theta \to 1$. For $Z = 0$ the momentum equation gives

$$\zeta''' + 3\zeta\zeta'' = 2(\zeta')^2 - 1 \quad (A1)$$

By inspection we see that for $\eta \to \infty$,

$$\eta' \to 1/\sqrt{2}.$$

This boundary condition is an asymptotic expression of the fact that for smaller Prandtl numbers (high thermal conductivities) high temperatures persist at larger distances from the wall and thus make the maximum velocity appear further and further away from the boundary. In the limit $Pr \to 0$, this maximum velocity will occur at an infinite distance. We define the transformation:

$$\zeta = \kappa\zeta_0(\eta_0), \quad \eta_0 = \lambda\eta. \quad (A2)$$

We calculate

$$\zeta' = \kappa\lambda\zeta_0'(\eta_0), \quad \zeta'' = \kappa\lambda^2\zeta_0''(\eta_0)$$
$$\zeta''' = \kappa\lambda^3\zeta_0'''(\eta_0). \quad (A3)$$

Substitution of the above in equation (A1) yields

$$\kappa\lambda^3\zeta_0''' + 3\kappa^2\lambda^2\zeta_0\zeta_0'' = (2\kappa^2\lambda^2\zeta_0')^2 - 1. \quad (A4)$$

We make

$$\kappa\lambda^3 = 3\kappa^2\lambda^2 \text{ and } 2\kappa^2\lambda^2 = 1.$$

From the above we deduce that

$$\kappa = \lambda/3 = (18)^{1/4} \text{ and } \kappa\lambda^3 = \tfrac{3}{2}.$$

With these values we can verify that for $\eta \to \infty$,

$$\zeta_0'(\eta_0) \to 1.$$

Now we have to solve the equation:

$$\zeta_0''' + \zeta_0\zeta_0'' = \tfrac{2}{3}\,[(\zeta_0')^2 - 1]. \quad \text{(A5)}$$

This is the well known solution for an incompressible flow over a wedge of angle $2\pi/3$ [10]. It is easy to calculate now that

$$\underset{\substack{Pr\to 0\\ Z=0}}{\zeta''(0)} = \frac{\sqrt{3}}{2^{3/4}}\,\zeta_0''(\eta_0 = 0) \approx 1{\cdot}07.$$

Résumé—Cet article traite le cas d'une plaque chaude verticale baignant dans un fluide électriquement conducteur sur lequel agit un champ magnétique perpendiculaire au mouvement induit dû aux forces de convection libre.

On a trouvé qu'il existait des solutions semblables pourvu que l'intensité du champ magnétique varie comme l'inverse de la racine quatrième de la distance mesurée dans la direction de l'écoulement.

Les équations différentielles du mouvement et de l'énergie ont des solutions qui dépendent du nombre de Prandtl, du nombre de Grashof et d'un troisième nombre sans dimensions (appelé Z) représentant le rapport des forces de pesanteur aux forces de convection libre.

Des solutions asymptotiques théoriques ont été obtenues pour une température de paroi constante dans les cas suivants:

(a) très grands nombres de Prandtl et petits nombres Z. Dans ce cas les forces d'inertie peuvent être négligées.

(b) très grands nombres Z par rapport aux nombres de Prandtl.

(c) petits nombres de Prandtl et nombre de Prandtl nul.

Des solutions exactes obtenues avec un calculateur analogique sont également rapportées. On a trouvé que le champ magnétique a pour effet de ralentir l'écoulement et par suite de diminuer le nombre de Nusselt. Pour un nombre de Prandtl constant, le taux de diminution du coefficient d'échange thermique est plus élevé pour des grands Z que pour des petits Z; d'autre part, pour une même valeur du paramètre Z, le taux de diminution de ce coefficient est plus grand pour des nombres de Prandtl plus bas.

Le cas des solutions non-semblables a également été étudié, les équations différentielles fondamentales sont données pour un champ magnétique transversal constant et pour des champs dépendant d'une puissance de la distance verticale.

On a trouvé que les expériences en laboratoire étaient possibles puisque le paramètre Z est de l'ordre de 1 à 10 pour les métaux liquides.

Zusammenfassung—Die Arbeit behandelt den Fall der beheizten senkrechten Platte in einer elektrisch leitenden Flüssigkeit und einem senkrecht zur Auftriebsrichtung wirkenden Magnetfeld. Ähnlichkeitslösungen können angegeben werden, wenn sich die Intensität des Magnetfeldes mit dem Reziprokwert der vierten Wurzel aus dem Abstand in Strömungsrichtung ändert. Die Lösungen der erhaltenen Differentialgleichungen für Bewegung und Energie hängen ab von der Prandtlzahl, der Grashofzahl und einer dritten dimensionslosen Grösse (genannt Z), die das Verhältnis der ponderomotorischen Kraft zur Auftriebskraft darstellt. Für folgende Fälle wurden bei konstanter Wandtemperatur theoretische asymptotische Lösungen gefunden:

(a) Sehr hohe Prandtl- und kleine Z-Zahlen. Die Trägheitskräfte können dabei vernachlässigt werden.

(b) Sehr hohe Z-Zahlen bei beliebigen Prandtlzahlen.

(c) Kleine Prandtlzahlen einschliesslich Null.

Ein Analogrechner lieferte exakte Lösungen. Es zeigte sich, dass das Magnetfeld den Fluss hemmte und damit die Nusseltzahl verkleinerte. Bei konstanter Prandtlzahl ist die Abnahme des Wärmeübergangskoeffizienten mit wachsendem Z grösser bei kleineren Z-Werten; bleibt der Parameter Z konstant, so erfolgt eine stärkere Abnahme des Wärmeübergangskoeffizienten bei kleineren Prandtlzahlen. Auch für den Fall, dass keine Ähnlichkeitslösung besteht, wurden Untersuchungen angestellt; die grundsätzlichen Differentialgleichungen für ein konstantes querlaufendes Magnetfeld und Felder, die vom senkrechten Abstand abhängen, sind angegeben. Da der Parameter Z für flüssige Metalle von der Grössenordnung 1/10 ist, sind Laboratoriumsversuche möglich.

Аннотация—Рассматривается случай для вертикальной нагретой пластины, окружённой электропроводной жидкостью при наличии магнитного поля, действующего в направлении перпендикулярном к движению, вызванному выталкивающими силами.

Найдено, что существуют подобные решения при условии, что интенсивность магнитного поля изменяется обратно пропорционально корню четвёртой степени расстояния, измеренного в направлении течения.

Полученные дифференциальные уравнения движения и энергии имеют решения, зависящие от числа Прандтля, числа Грасгофа и безразмерного числа (названного Z), представляющего собой соотношение пондермоторной и выталкивающей сил.

Получены асимптотические решения для постоянной температуры стенки в случаях:

(а) число Прандтля весьма велико, а Z, мало. В этом случае можно пренебречь силами инерции.

(б) Z весьма велико, а число Прандтля не учитывается;

(в) при нулевом и весьма малом значении числа Прандтля.

Приводятся точные решения, полученные численным методом на вычислительной машине. Найдено, что влияние магнитного поля сказывается в торможении потока и, следовательно, в уменьшении числа Нуссельта. При постоянном числе Прандтля скорость уменьшения коэффициента теплообмена убывает с увеличением Z; с другой стороны, при неизменном Z скорость уменьшения коэффициента теплообмена выше для малых чисел Прандтля.

Рассматривается также случай неавтомодельных решений. Даются также основные дифференциальные уравнения для постоянного поперечного магнитного поля и в случае степенной зависимости коэффициента индукции от расстояния по вертикали.

Высказывается соображение о возможности экспериментальной проверки полученных зависимостей лабораторным путём, поскольку параметр Z имеет порядок величины от 1 до 10 для жидких металлов.